

IFUM-684-FT
UTHET-01-0401
hep-th/0104141

Universality and a generalized C -function in CFTs with AdS Duals

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Abstract

We argue that the thermodynamics of conformal field theories with AdS duals exhibits a remarkable universality. At strong coupling, a Cardy-Verlinde entropy formula holds even when R -charges or bulk supergravity scalars are turned on. In such a setting, the Casimir entropy can be identified with a generalized C -function that changes monotonically with temperature as well as when non-trivial bulk scalar fields are introduced. We generalize the Cardy-Verlinde formula to cases where no subextensive part of the energy is present and further observe that such a formula is valid for the $\mathcal{N} = 4$ super Yang-Mills theory in $D = 4$ even at weak coupling. Finally we show that a generalized Cardy-Verlinde formula holds for asymptotically flat black holes in any dimension.

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1 Introduction

Recently, the thermodynamics of conformal field theories (CFTs) with gravity duals has attracted renewed interest. Much of it is due to the remarkable resemblance of the relevant thermodynamical formulae to standard cosmology [1]. This has led to interesting conjectures regarding cosmological scenarios that have been exploited in a number of works [2].

Apart from cosmology the crucial observations in [1] should have implications for the thermodynamics of CFTs *per se* [3]. In a previous work [4] we made a step in that direction by studying general entropy bounds in weakly and strongly coupled CFTs. The starting point of our investigation was the Cardy-Verlinde entropy formula

$$S = \frac{2\pi R}{n} \sqrt{E_C(2E - E_C)}, \quad (1.1)$$

for CFTs on $\mathbb{R} \times S^n$, where R denotes the radius of the S^n and E_C is the Casimir energy defined in [1]. The above formula has been shown to hold in a large number of strongly coupled CFTs whose thermodynamics is described by higher-dimensional supergravity solutions, e. g. Kerr-AdS black holes with one rotation parameter [4], charged black holes in several gauged supergravity theories [5], or Taub-Bolt-AdS spacetimes [6]. Whether a similar formula is valid for weakly coupled CFTs is not completely clear as yet. In fact the deeper origin of (1.1) remains rather obscure, primarily because the original Cardy formula [7] follows from modular invariance which is a characteristic feature of two-dimensional conformal field theories only.

The main purpose of the present work is to study further the above issues. In particular, following [4] we will argue that the Cardy-Verlinde formula is the outcome of a striking resemblance between the thermodynamics of CFTs with AdS duals and CFTs in two dimensions. We will also argue that the Casimir entropy may be viewed as a generalized C -function since it exhibits a monotonic behavior under temperature changes as well as in cases when a holographic RG-flow is induced by bulk scalars in the boundary CFT. Such an interpretation allows us to generalize (1.1) to the case where no subextensive part of the energy is present, i. e. when the Casimir energy vanishes but the Casimir entropy is different from zero. Moreover we show that (1.1) is valid also for weakly coupled $D = 4$, $\mathcal{N} = 4$ super Yang-Mills theory, with an overall coefficient $4\pi R/3$ instead of $2\pi R/3$.

The paper is organized as follows. In Section 2 we study the thermodynamics of black hole solutions in gauged supergravities with non-trivial bulk scalar fields. In all cases considered we find that (1.1) is valid for the corresponding dual CFTs. In Section 3 we discuss the validity of (1.1) for CFTs in flat spaces, e. g. dual to black holes with flat horizons. This allows us to discuss the monotonicity properties of the Casimir entropy under temperature changes as well as under the switching on of bulk scalars. Our results suggest that the Casimir entropy can be interpreted as a monotonic, generalized C -function for theories at finite temperature. In Section 4 we show that the Cardy-Verlinde formula (1.1) remarkably holds also for the weakly coupled $\mathcal{N} = 4$ SYM in $D = 4$, with an overall coefficient which is twice the one in the strongly coupled limit. Finally, in Section 5 we

show that a generalized Cardy formula holds also for asymptotically flat black holes in any dimension. We conclude and present an outlook of our ideas in Section 6.

2 Black holes in gauged supergravity and the Cardy-Verlinde entropy formula

We begin by considering charged black holes in diverse gauged supergravities studied in [5]. The STU model of $D = 5$, $N = 2$ gauged supergravity¹ admits the black hole solutions [8, 9]

$$ds^2 = -(H_1 H_2 H_3)^{-2/3} f dt^2 + (H_1 H_2 H_3)^{1/3} (f^{-1} dr^2 + r^2 d\Omega_3^2), \quad (2.1)$$

where

$$f = 1 - \frac{\mu}{r^2} + r^2 R^{-2} H_1 H_2 H_3, \quad H_I = 1 + \frac{q_I}{r^2}, \quad I = 1, 2, 3. \quad (2.2)$$

The moduli X^I and the gauge potentials A^I are given by

$$X^I = H_I^{-1} (H_1 H_2 H_3)^{1/3}, \quad A_t^I = \frac{\tilde{q}_I}{r^2 + q_I}, \quad (2.3)$$

where the \tilde{q}_I denote the physical charges related to the q_I by

$$q_I = \mu \sinh^2 \beta_I, \quad \tilde{q}_I = \mu \sinh \beta_I \cosh \beta_I. \quad (2.4)$$

The BPS limit [10] is reached when the nonextremality parameter μ goes to zero and $\beta_I \rightarrow \infty$, with q_I fixed². The horizon coordinate r_+ obeys

$$\mu = r_+^2 \left(1 + \frac{1}{R^2 r_+^4} \prod_{I=1}^3 (r_+^2 + q_I) \right). \quad (2.5)$$

The black hole mass and the Bekenstein-Hawking entropy read, respectively

$$\begin{aligned} M &= \frac{\pi}{4G} \left(\frac{3}{2} \mu + \sum_I q_I \right), \\ S &= \frac{\pi^2}{2G} \sqrt{\prod_I (r_+^2 + q_I)}. \end{aligned} \quad (2.6)$$

We define the excitation energy above the BPS state,

$$E = M - M_{BPS} = \frac{3\pi}{8G} \mu = \frac{3\pi}{8G} r_+^2 \left[1 + \frac{1}{R^2 r_+^4} \prod_I (r_+^2 + q_I) \right]. \quad (2.7)$$

¹This model can also be embedded into $D = 5$, $N = 8$ gauged supergravity.

²The reader should note that the BPS limit represents naked singularities.

Following [4] we define a parameter Δ by

$$\Delta^{-2} = \frac{1}{R^2 r_+^4} \prod_I (r_+^2 + q_I). \quad (2.8)$$

This yields

$$2ER = \frac{3}{2\pi} S\Delta [1 + \Delta^{-2}], \quad (2.9)$$

which is exactly the behavior of a two-dimensional CFT with characteristic scale R , temperature $\tilde{T} = 1/(2\pi R\Delta)$, and central charge proportional to $S\Delta$. This resemblance motivates to define the Casimir energy as the subextensive part of (2.9)

$$E_C R = \frac{3}{2\pi} S\Delta. \quad (2.10)$$

Now one easily verifies that the quantities (2.6), (2.7) and (2.10) satisfy exactly the Cardy-Verlinde formula (1.1) for $n = 3$. As in [1] we can define the Casimir entropy S_C by

$$S_C = \frac{2\pi}{n} E_C R = S\Delta. \quad (2.11)$$

In terms of S_C , (2.9) can be rewritten as

$$2ER = \frac{3}{2\pi} S_C [1 + \Delta^{-2}], \quad (2.12)$$

which allows to interpret the Casimir entropy as a generalization of the central charge to higher dimensions. This interpretation of S_C was already pointed out by Verlinde [1].

The above considerations can be easily generalized to black holes in $D = 4$ and $D = 7$ gauged supergravities as well. That (1.1) holds in these cases was shown in [5], but we would like to reformulate the results of [5] in a different way, in order to stress that the validity of (1.1) can be traced back to an effective two-dimensional behavior.

Black hole solutions in a truncated version of $D = 4$, $N = 8$ gauged supergravity have been found in [12]. The metric is given by

$$ds^2 = -(H_1 H_2 H_3 H_4)^{-1/2} f dt^2 + (H_1 H_2 H_3 H_4)^{1/2} (f^{-1} dr^2 + r^2 d\Omega_2^2), \quad (2.13)$$

where

$$f = 1 - \frac{\mu}{r} + R^{-2} r^2 \prod_{I=1}^4 H_I, \quad H_I = 1 + \frac{q_I}{r}, \quad I = 1, \dots, 4. \quad (2.14)$$

Furthermore there are gauge fields and scalars turned on (cf. [12] for details). The nonextremality parameter μ is given in terms of the horizon radius r_+ by

$$\mu = r_+ \left(1 + \frac{1}{l^2 r_+^2} \prod_I (r_+ + q_I) \right). \quad (2.15)$$

The mass and entropy read

$$\begin{aligned} M &= \frac{1}{4G}(2\mu + \sum_I q_I), \\ S &= \frac{\pi}{G} \sqrt{\prod_I (r_+ + q_I)}, \end{aligned} \quad (2.16)$$

respectively. Again we define the excitation energy $E = M - M_{BPS}$ above the BPS state (which is the one with $\mu = 0$), yielding

$$2ER = \frac{r_+ R}{G} \left[1 + \frac{1}{R^2 r_+^2} \prod_I (r_+ + q_I) \right]. \quad (2.17)$$

Defining also

$$\Delta^{-2} = \frac{1}{R^2 r_+^2} \prod_I (r_+ + q_I), \quad (2.18)$$

we obtain

$$2ER = \frac{1}{\pi} S \Delta [1 + \Delta^{-2}], \quad (2.19)$$

which resembles again the behavior of a two-dimensional CFT. The Casimir energy is then easily determined from (2.19) as the subextensive part, i. e.

$$E_C R = \frac{1}{\pi} S \Delta = \frac{1}{\pi} S_C. \quad (2.20)$$

The quantities S , E and E_C satisfy again the Cardy-Verlinde formula (1.1) for $n = 2$.

Two-charge black hole solutions of $D = 7$, $N = 4$ gauged supergravity can be found in [13, 9]. We give here only the metric, which reads

$$ds^2 = -(H_1 H_2)^{-4/5} f dt^2 + (H_1 H_2)^{1/5} (f^{-1} dr^2 + r^2 d\Omega_5^2), \quad (2.21)$$

with

$$f(r) = 1 - \frac{\mu}{r^4} + r^2 R^{-2} H_1 H_2, \quad H_I = 1 + \frac{q_I}{r^4}, \quad I = 1, 2. \quad (2.22)$$

The nonextremality parameter μ is related to the black hole horizon r_+ by

$$\mu = r_+^4 + \frac{1}{r_+^2 l^2} \prod_I (r_+^4 + q_I). \quad (2.23)$$

The mass and entropy are

$$\begin{aligned} M &= \frac{\pi^2}{4G} \left(\frac{5}{4}\mu + \sum_I q_I \right), \\ S &= \frac{\pi^3 r_+}{4G} \sqrt{\prod_I (r_+^4 + q_I)}. \end{aligned} \tag{2.24}$$

Defining as above

$$E = M - M_{BPS} = M - M_{\mu=0}, \tag{2.25}$$

and

$$\Delta^{-2} = \frac{1}{R^2 r_+^6} \prod_I (r_+^4 + q_I), \tag{2.26}$$

one obtains

$$2ER = \frac{5}{2\pi} S \Delta [1 + \Delta^{-2}] = \frac{5}{2\pi} S_C [1 + \Delta^{-2}], \tag{2.27}$$

so that $E_C R = (5/2\pi) S_C$, and (1.1) still holds.

The above considerations go through also for Reissner-Nordström-AdS black holes in arbitrary dimension $n + 2$,

$$\begin{aligned} ds^2 &= -H^{-2} f dt^2 + H^{\frac{2}{n-1}} (f^{-1} dr^2 + r^2 d\Omega_n^2), \\ H &= 1 + \frac{q}{r^{n-1}}, \\ f &= 1 - \frac{\mu}{r^{n-1}} + r^2 R^{-2} H^{\frac{2n}{n-1}}, \end{aligned}$$

with mass and entropy given by

$$\begin{aligned} M &= \frac{n V_n}{16\pi G} (\mu + 2q), \\ S &= \frac{V_n}{4G} (r_+^{n-1} + q)^{\frac{n}{n-1}}, \end{aligned}$$

where V_n denotes the volume of the unit n -sphere. With $E = M - M_{\mu=0}$ one finds

$$2ER = \frac{n}{2\pi} S_C [1 + \Delta^{-2}], \tag{2.28}$$

where

$$\Delta^{-2} = \frac{1}{R^2 r_+^{2(n-1)}} (r_+^{n-1} + q)^{\frac{2n}{n-1}}, \tag{2.29}$$

and the central charge $S_C = S\Delta$. This leads again to (1.1).

In order to emphasize the complete universality of (1.1) and (2.28), let us consider as a final example the Kerr-AdS black hole in five dimensions with two rotation parameters. The metric reads [14]

$$\begin{aligned}
ds^2 = & -\frac{\Delta_r}{\rho^2} \left(dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right)^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left(a dt - \frac{(r^2 + a^2)}{\Xi_a} d\phi \right)^2 \\
& + \frac{\Delta_\theta \cos^2 \theta}{\rho^2} \left(b dt - \frac{(r^2 + b^2)}{\Xi_b} d\psi \right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 \\
& + \frac{(1 + r^2 R^{-2})}{r^2 \rho^2} \left(ab dt - \frac{b(r^2 + a^2) \sin^2 \theta}{\Xi_a} d\phi - \frac{a(r^2 + b^2) \cos^2 \theta}{\Xi_b} d\psi \right)^2,
\end{aligned} \tag{2.30}$$

where

$$\begin{aligned}
\Delta_r &= \frac{1}{r^2} (r^2 + a^2)(r^2 + b^2)(1 + r^2 R^{-2}) - 2m; \\
\Delta_\theta &= (1 - a^2 R^{-2} \cos^2 \theta - b^2 R^{-2} \sin^2 \theta); \\
\rho^2 &= (r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta); \\
\Xi_a &= (1 - a^2 R^{-2}); \quad \Xi_b = (1 - b^2 R^{-2}).
\end{aligned} \tag{2.31}$$

The horizon location r_+ is the largest root of $\Delta_r = 0$. The mass and the entropy are given by [14]

$$\begin{aligned}
M &= \frac{3\pi}{8\Xi_a \Xi_b G r_+^2} (r_+^2 + a^2)(r_+^2 + b^2)(1 + r_+^2 R^{-2}), \\
S &= \frac{\pi^2}{2r_+ \Xi_a \Xi_b G} (r_+^2 + a^2)(r_+^2 + b^2).
\end{aligned}$$

With $E = M$, $\Delta^{-1} = r_+/R$ and $S_C = S\Delta$ we can write

$$2ER = \frac{3}{2\pi} S_C [1 + \Delta^{-2}], \tag{2.32}$$

and thus also in the rotating case the two-dimensional CFT behavior persists, and the Cardy-Verlinde formula (1.1) is satisfied.

3 Universality and monotonicity properties of the generalized C -function

The results obtained in the previous Section reveal a remarkable universality in the thermodynamics of CFTs with AdS duals. The supergravity theories studied above contain

bulk gauge fields that couple to R -currents on the boundary and therefore the dual CFTs have R -charges turned on. Also, they contain bulk scalar fields that either couple to operators in the boundary CFT which acquire nonvanishing expectation values, or else induce RG-flows [15]. In all cases we found a striking resemblance of the thermodynamics of the boundary CFT to the one of a CFT in two dimensions. The only parameter that apparently encodes the properties of the CFT under consideration is Δ . The same universal behavior is observed also for rotating black holes which are dual to strongly coupled CFTs on a rotating Einstein universe [4].

An important ingredient in our calculations has been the fact that we were able to define the Casimir entropy S_C in such a way that it closely resembles a two-dimensional central charge. We would like to substantiate this observation. At high temperatures one easily shows that S_C is proportional to the derivative of the entropy with respect to the temperature. This means that it is related to the number of degrees of freedom between T and $T + dT$, hence it lends itself as a good candidate for a generalized C -function in theories at finite temperature. For such an interpretation to be meaningful, one should at least demonstrate that S_C possesses certain monotonicity properties. For the general cases considered in Section 2, S_C is a function of *both* the temperature and the R -charges. Then it is crucial to study the behavior of S_C *both* under temperature changes as well as when non-trivial bulk fields are turned on. The latter case should be related to the behavior of the Casimir entropy under RG-flows, in much the same way as the presence of non-trivial fields in the bulk theory induces a holographic RG-flow in the boundary theory.

In order to analyze the behavior of the Casimir entropy under temperature changes we consider the field theory dual to the Schwarzschild-AdS black hole in $n + 2$ dimensions. In this case one has [4]

$$S_C = \frac{V_n R}{4G} r_+^{n-1}. \quad (3.1)$$

One easily finds that the monotonicity property

$$T \frac{dS_C}{dT} \geq 0, \quad (3.2)$$

is equivalent to $r_+^2/R^2 \geq (n-1)/(n+1)$, and thus coincides with the region of stability above the Hawking-Page phase transition [16] where the free energy is a concave function of the temperature. This indicates that there is a direct relationship between monotonicity of the generalized C -function and local thermodynamic stability.

Next we want to study how S_C changes when non-trivial bulk scalars are turned on. In many cases, turning on bulk scalar has been shown to induce an RG-flow in the dual theory [15]. Since such holographic RG-flows have been studied for dual field theories that live in flat space, the first step is to understand how to generalize the Cardy-Verlinde formula and the definition of the Casimir entropy for such cases, e.g. when no subextensive part of the energy is present. Let us consider the AdS black holes in dimension $n + 2$ with

flat horizon and metric given by

$$\begin{aligned} ds^2 &= -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\vec{x}^2, \\ f(r) &= -\frac{2m}{r^{n-1}} + \frac{r^2}{R^2}, \end{aligned} \quad (3.3)$$

where $d\vec{x}^2$ denotes the line element on a flat n -dimensional Euclidean space. The mass and entropy densities read

$$\begin{aligned} \frac{E}{V} &= \frac{n}{16\pi G} \frac{r_+^{n+1}}{R^2}, \\ \frac{S}{V} &= \frac{r_+^n}{4G}, \end{aligned}$$

where r_+ is the largest root of $f(r) = 0$. In two-dimensional conformal field theories the Cardy formula

$$S = 2\pi \sqrt{\frac{cL_0}{6}} \quad (3.4)$$

is valid only when the conformal weight of the ground state is zero, otherwise (in the presence of a Casimir energy) one has to modify (3.4) to

$$S = 2\pi \sqrt{\frac{c}{6} \left(L_0 - \frac{c}{24} \right)}, \quad (3.5)$$

which is the two-dimensional form of (1.1). For the black holes (3.3), the dual CFT lives on a flat space, and thus the energy has no subextensive part. Since the Casimir energy vanishes, the Cardy-Verlinde formula (1.1) makes no sense in this case. However, we saw above that the appropriate generalization of the central charge C to higher dimensions is essentially given by the Casimir entropy. If we use $E_C R = (n/2\pi)S_C$ in (1.1), and drop the subtraction of E_C in analogy with (3.4), we obtain the generalization of (3.4) to $n+1$ dimensions,

$$S = \frac{2\pi}{n} \sqrt{\frac{cL_0}{6}}, \quad (3.6)$$

where $L_0 = ER$ and $c/6 = nS_C/\pi$. Using $\Delta^{-1} = r_+/R$, one easily verifies that the black holes (3.3) satisfy the modified Cardy-Verlinde formula (3.6).

Now we are ready to proceed. A suitable class of models that exhibit both holographic RG-flows and non-zero temperature are the STU black holes (2.1) with flat horizons [8, 9], given by³

$$\begin{aligned} ds^2 &= -(H_1 H_2 H_3)^{-2/3} f dt^2 + (H_1 H_2 H_3)^{1/3} (f^{-1} dr^2 + r^2 d\vec{x}^2), \\ X^I &= H_I^{-1} (H_1 H_2 H_3)^{1/3}, \quad A_t^I = \frac{\sqrt{q_I \mu}}{r^2 + q_I}, \end{aligned} \quad (3.7)$$

³Note that this solution can be obtained from (2.1) by writing the metric on S^3 as $d\Omega_3^2 = d\chi^2 + \sin^2 \chi d\Omega_2^2$, and then performing the scaling limit $\chi \rightarrow \epsilon\chi$, $r \rightarrow r/\epsilon$, $t \rightarrow \epsilon t$, $\mu \rightarrow \mu/\epsilon^4$, $\beta_I \rightarrow \epsilon\beta_I$, $\epsilon \rightarrow 0$.

where

$$f = -\frac{\mu}{r^2} + r^2 R^{-2} H_1 H_2 H_3, \quad (3.8)$$

and the harmonic functions are given in (2.2). In [11] it has been shown that, in the extremal limit, the presence of non-trivial⁴ bulk scalar fields gives rise to a holographic RG-flow. Indeed, when $\mu = 0$ the metric (3.7) can be put in the domain wall form

$$ds^2 = e^{2A(\rho)}[-dt^2 + d\vec{x}^2] + d\rho^2, \quad (3.9)$$

with

$$e^{2A(\rho)} = \frac{r^2}{R^2} (H_1 H_2 H_3)^{\frac{1}{3}}, \quad \frac{dr}{d\rho} = \frac{r}{R} (H_1 H_2 H_3)^{\frac{1}{3}}. \quad (3.10)$$

Then one computes the C -function [15] which describes the induced RG-flow as⁵

$$C \sim \frac{1}{G[A'(\rho)]^3} = \frac{(3R)^3}{G} \frac{(H_1 H_2 H_3)^2}{(H_1 H_2 + H_1 H_3 + H_2 H_3)^3}. \quad (3.11)$$

The UV limit is reached when $r \rightarrow \infty$ and the IR limit when $r \rightarrow 0$. Equivalently, the UV limit is reached as $q_1, q_2, q_3 \rightarrow 0$ and the IR limit as $q_1, q_2, q_3 \rightarrow \infty$. The corresponding expressions for the C -function read then

$$C^{UV} \sim \frac{R^3}{G} > C^{IR} \sim \frac{(3R)^3}{G} \frac{(q_1 q_2 q_3)^2}{(q_1 q_2 + q_1 q_3 + q_2 q_3)^3}. \quad (3.12)$$

In the example above, the extremal limit leads to a field theory at zero temperature and one obtains the usual generalized C -function defined from the holographic RG-flow of theories which admit dual gravitational descriptions.

Away from the extremal limit we obtain a dual field theory at finite temperature. We want now to show that also at finite temperature we can implement an RG-flow. We argue as follows. For the metric (3.7), the mass and Bekenstein-Hawking entropy are

$$E = \frac{3V_3}{16\pi G} \mu = \frac{3V_3}{16\pi G} \frac{1}{R^2 r_+^2} \prod_I (r_+^2 + q_I), \quad (3.13)$$

$$S = \frac{V_3}{4G} \sqrt{\prod_I (r_+^2 + q_I)}. \quad (3.14)$$

These are interpreted as the mass and the entropy of the corresponding dual CFT. With Δ^{-2} defined in (2.8), one gets for the Casimir entropy

$$S_C = S\Delta = \frac{V_3 R r_+^2}{4G}. \quad (3.15)$$

⁴It was shown in [11] that one requires $q_1 \neq q_2 \neq q_3$ for non-singular flows.

⁵Here the 5-dimensional Newton constant G satisfies the AdS/CFT relation $\frac{2N^2}{\pi} = \frac{R^3}{G}$.

One easily checks that (3.13), (3.14) and (3.15) satisfy (3.6) with $L_0 = ER$ and $c/6 = 3S_C/\pi$. Notice now that since the horizon coordinate obeys

$$\mu = \frac{1}{R^2 r_+^2} \prod_{I=1}^3 (r_+^2 + q_I), \quad (3.16)$$

the value of S_C in (3.15) depends implicitly on the charges q_I . Following up with our proposal to interpret S_C as a generalized C -function, we keep the effective temperature of the dual CFT fixed and allow only the variation of the q_I . Now, as we have done in the zero temperature limit, we want to see how S_C changes in the two limits $q_I \rightarrow 0, \infty$, which were the UV and IR limits respectively of the CFT dual to the extremal case. Then, by virtue of (3.16) we see that under such a variation S_C changes monotonically. More specifically in the UV limit where $q_I \rightarrow 0$, which is equivalent to taking r_+ large, we obtain the usual result for the Casimir entropy of $\mathcal{N} = 4$ SYM at strong coupling

$$S_C^{UV} = \frac{V_3 R^3}{4G} \left(\frac{r_+^0}{R} \right)^2, \quad (3.17)$$

where now r_+^0 denotes the solution of (3.16) for $q_I = 0$. In the IR limit where $q_I \rightarrow \infty$, or equivalently for small r_+ ($r_+^2 \ll q_I$), we obtain

$$S_C^{IR} = S_C^{UV} \frac{r_+^3}{\sqrt{q_1 q_2 q_3}} \ll S_C^{UV}. \quad (3.18)$$

Using the same approach as above, we moreover find a quite interesting RG-flow for a boundary CFT that exhibits a temperature phase transition. This is the case of the CFT dual to the STU black hole with spherical horizons. It is described by the metric

$$ds^2 = -(H_1 H_2 H_3)^{-2/3} f dt^2 + (H_1 H_2 H_3)^{1/3} (f^{-1} dr^2 + r^2 d\Omega_3^2), \quad (3.19)$$

where

$$f = 1 - \frac{\mu}{r^2} + r^2 R^{-2} H_1 H_2 H_3. \quad (3.20)$$

The horizon coordinate now obeys

$$\mu = r_+^2 \left(1 + \frac{1}{R^2 r_+^4} \prod_{I=1}^3 (r_+^2 + q_I) \right), \quad (3.21)$$

Notice that a solution exists only if

$$\mu R^2 > q_1 q_2 + q_2 q_3 + q_3 q_1. \quad (3.22)$$

To relate our discussion with the RG-flow described before, we wish to consider the extremal limit $\mu \rightarrow 0$. This, however, can only be achieved if at the same time we let $R \rightarrow \infty$. From (3.20) it is natural to take

$$\mu \gtrsim \frac{\sqrt{q_1 q_2 q_3}}{R}. \quad (3.23)$$

The Casimir entropy is still given by (3.15). An important property of the theory dual to (3.19) is that it exhibits a Hawking-Page phase transition at a temperature defined by the relation $\Delta = 1$. This is exactly the point where the following entropy bound

$$\frac{S}{S_B} = \frac{2\Delta^{-1}}{1 + \Delta^{-2}} \leq 1, \quad S_B = \frac{2\pi ER}{3}, \quad (3.24)$$

is saturated. As discussed before, at this special point the flow of S_C induced by the non-trivial moduli is monotonic. Specifically, for large r_+ (large black hole), we have $r_+ = R$ due to $R^2 \gg q_I$. Therefore we obtain

$$S_C = S = \frac{\pi^2}{2G} R^3, \quad (3.25)$$

which corresponds to the $\mathcal{N} = 4$ Super Yang-Mills model. For small r_+ , in which case $Rr_+^2 = \sqrt{q_1 q_2 q_3}$, we find

$$S_C = \frac{\pi^2}{2G} \sqrt{q_1 q_2 q_3}. \quad (3.26)$$

We can now see that (3.26) corresponds to a BPS state. This can be obtained from a D1-brane of charge $Q_1 \sim q_1/\sqrt{G}$, together with a D5-brane of charge $Q_5 \sim q_2/\sqrt{G}$ in six-dimensions [17]. By compactifying the sixth dimension along a circle of radius R_0 , we obtain momenta $P \sim N/R_0$, where $N \sim q_3/G$. Then a Kaluza-Klein reduction produces a black hole with charges Q_1 , Q_5 and N , and entropy

$$S \sim \sqrt{Q_1 Q_5 N} \sim \frac{1}{G} \sqrt{q_1 q_2 q_3} \quad (3.27)$$

The examples studied above suggest that indeed the Casimir entropy S_C is a natural candidate for a generalized C -function for field theories that have a dual supergravity description, both at zero and at finite temperature. In general S_C is a function of *both* the temperature and the renormalization scale (which enters the game through the warp-factor $A(\rho)$ in (3.9)) of the field theory. We have found that S_C is a monotonic function in each variable, while keeping the other one fixed. At fixed energy scale we move in the space of conformally invariant theories at varying temperature, while at fixed temperature we move along the renormalization group trajectories.

4 Free field theory side

Now we go back to manifolds of the form $\mathbb{R} \times S^n$, and start with a system of N_B scalars, N_F Weyl fermions and N_V vectors on $\mathbb{R} \times S^3$, with the radius of the S^3 given by R . The free energy has been computed in [18, 3], and reads

$$-FR = a_4 \delta^{-4} + a_2 \delta^{-2} + a_0, \quad (4.1)$$

where $\delta^{-1} = 2\pi RT$ and

$$\begin{aligned} a_4 &= \frac{1}{720}(N_B + \frac{7}{4}N_F + 2N_V), & a_2 &= -\frac{1}{24}(\frac{1}{4}N_F + 2N_V), \\ a_0 &= \frac{1}{240}(N_B + \frac{17}{4}N_F + 22N_V), \end{aligned} \quad (4.2)$$

satisfying the constraint $3a_4 = a_2 + a_0$. The entropy and energy are, respectively,

$$S = 2\pi(4a_4\delta^{-3} + 2a_2\delta^{-1}), \quad ER = 3a_4\delta^{-4} + a_2\delta^{-2} - a_0. \quad (4.3)$$

One thus obtains

$$2ER = \frac{1}{2\pi}S\delta[1 + \delta^{-2}]\frac{3a_4 - a_0\delta^2}{2a_4 + a_2\delta^2}. \quad (4.4)$$

Remarkably, for the $\mathcal{N} = 4$ SYM model, this simplifies to

$$2ER = \frac{3}{4\pi}S\delta[1 + \delta^{-2}], \quad (4.5)$$

which resembles again the behavior of a two-dimensional CFT with central charge given by $S_C \equiv S\delta$. We can define the Casimir energy as the subextensive part of (4.5), i. e.

$$E_C R = \frac{3}{4\pi}S\delta. \quad (4.6)$$

It is then easily shown that a Cardy-Verlinde formula

$$S = \frac{4\pi R}{3}\sqrt{E_C(2E - E_C)} \quad (4.7)$$

holds. Note that the prefactor in (4.7) is twice the one in (1.1), which is valid at strong 't Hooft coupling. Note also that the parameter δ at weak coupling is different from Δ used at strong coupling. Whereas δ^{-1} is related to the true temperature, Δ^{-1} represents an effective temperature [4], which, for $T \rightarrow \infty$, becomes again directly related to T .

We thus found that for the $\mathcal{N} = 4$ SYM model on $\mathbb{R} \times S^3$, a generalized Cardy formula

$$S = \frac{b\pi R}{3}\sqrt{E_C(2E - E_C)} \quad (4.8)$$

holds, with $b = 2$ for strong coupling and $b = 4$ for free fields. The fact that (4.8) is valid in both extremal regimes suggests that it may hold for every value of the coupling, with b being a function of $g_{YM}^2 N$, that interpolates smoothly (at least for temperatures above the Hawking-Page phase transition) between the values 2 for $g_{YM}^2 N \rightarrow \infty$ and 4 for $g_{YM}^2 N \rightarrow 0$. As a first step to see whether such a conjecture makes sense, one would wish to compute the leading stringy corrections to the supergravity approximation, in order to see if they are indeed positive. In the low energy effective action, massive string modes

manifest themselves as higher derivative curvature terms. In type IIB supergravity, the lowest correction is of order $\alpha'^3 R^4$ [19], where R denotes the Riemann tensor. The leading stringy corrections to the thermodynamics of Schwarzschild-AdS black holes (without rotation or charges) have been computed in [20, 21], generalizing the calculations for SYM on flat space [22, 23]. As we said, we expect the relation

$$2ER = \frac{3}{b\pi} S\Delta[1 + \Delta^{-2}] \quad (4.9)$$

to be completely universal, i. e. to hold also for the α'^3 -corrected thermodynamical quantities. The problem is now that in (4.9) one has two unknown functions, namely b and Δ . In fact, for the free field theory $\Delta^{-1} = \delta^{-1} = 2\pi RT$, whereas for strong coupling $\Delta^{-1} = r_+/R$ is a complicated function of the temperature. It is thus clear that also Δ changes with the coupling. One might be tempted to set $\tilde{\Delta} = R/\tilde{r}_+$ for the α'^3 -corrected function, where \tilde{r}_+ is the corrected horizon location, which is known [21]. This choice is however not obvious, because in principle $\tilde{\Delta}$ could be a more complicated expression in terms of \tilde{r}_+ . Note also that at strong coupling one has $E_C = (n+1)E - nTS$ with $n = 3$, whereas for free fields, (4.6) satisfies $E_C = (4E - 3TS)/2$. We will leave the further study of stringy corrections to the Cardy-Verlinde formula for a future publication.

5 A generalized Cardy-Verlinde formula for asymptotically flat black holes

All cases where the Cardy-Verlinde formula has been shown to hold up to now had as a necessary ingredient a negative cosmological constant, or, more generally, a certain potential term for supergravity scalars. This guarantees that the theory admits AdS vacua, and thus one has a dual description in terms of a conformal field theory on the boundary of AdS. A natural question is whether the Cardy-Verlinde formula holds in a more general setting, e. g. for black holes that are asymptotically flat rather than approaching AdS space. We will show that this is indeed the case. Let us first consider the Schwarzschild solution in $n+2$ dimensions, given by

$$ds^2 = - \left(1 - \frac{2m}{r^{n-1}}\right) dt^2 + \left(1 - \frac{2m}{r^{n-1}}\right)^{-1} dr^2 + r^2 d\Omega_n^2. \quad (5.1)$$

The mass, entropy and temperature read

$$E = \frac{nV_n}{16\pi G} r_+^{n-1}, \quad S = \frac{V_n}{4G} r_+^n, \quad T = \frac{n-1}{4\pi r_+}, \quad (5.2)$$

where r_+ is the horizon location obeying $r_+^{n-1} = 2m$. As in [1], we can now define the Casimir energy as the violation of the Euler identity,

$$E_C = n(E - TS + pV). \quad (5.3)$$

Let us assume for the moment that, like their Schwarzschild-AdS counterparts, also the black holes (5.1) in flat space are described by a conformal field theory in $n+1$ dimensions. Then the stress tensor is traceless, which implies the equation of state $pV = E/n$. Using this in (5.3), one gets

$$E_C = (n+1)E - nTS. \quad (5.4)$$

With (5.2), we thus obtain

$$E_C = \frac{nV_n}{8\pi G} r_+^{n-1} \quad (5.5)$$

for the Schwarzschild black hole. Note that, due to $E_C = 2E$, the Cardy-Verlinde formula (1.1) would yield $S = 0$, which suggests that one has to use instead the generalization (3.6) of the usual Cardy formula (3.4), which is valid if the ground state has zero conformal weight. As the only scale appearing in the metric (5.1) is given by the Schwarzschild radius r_+ , an obvious generalization of (1.1) would be

$$S = \frac{2\pi r_+}{n} \sqrt{E_C \cdot 2E}. \quad (5.6)$$

One easily verifies that the Casimir energy (5.5) and the thermodynamical quantities (5.2) satisfy this modified Cardy-Verlinde formula.

As a confirmation of (5.6), we consider the asymptotically flat Kerr black holes [24]⁶

$$\begin{aligned} ds^2 = & -\frac{\Delta_r}{\rho^2} [dt - a \sin^2 \theta d\phi]^2 + \frac{\rho^2}{\Delta_r} dr^2 + \rho^2 d\theta^2 \\ & + \frac{\sin^2 \theta}{\rho^2} [adt - (r^2 + a^2)d\phi]^2 + r^2 \cos^2 \theta d\Omega_{n-2}^2, \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} \Delta_r &= (r^2 + a^2) - 2mr^{3-n}, \\ \rho^2 &= r^2 + a^2 \cos^2 \theta. \end{aligned} \quad (5.8)$$

The inverse temperature, free energy, entropy, energy, angular momentum and angular velocity of the horizon read [24]

$$\beta = \frac{4\pi(r_+^2 + a^2)}{(n-1)r_+ + \frac{(n-3)a^2}{r_+}}, \quad F = \frac{V_n}{16\pi G} r_+^{n-3} (r_+^2 + a^2), \quad (5.9)$$

$$S = \frac{V_n}{4G} r_+^{n-2} (r_+^2 + a^2), \quad E = \frac{nV_n}{16\pi G} r_+^{n-3} (r_+^2 + a^2), \quad (5.10)$$

$$J = \frac{aV_n}{8\pi G} r_+^{n-3} (r_+^2 + a^2), \quad \Omega = \frac{a}{r_+^2 + a^2}, \quad (5.11)$$

⁶For simplicity we restrict ourselves to the case of only one rotation parameter.

where r_+ denotes the largest root of $\Delta_r = 0$. We can again define the Casimir energy as the violation of the Euler identity, i. e.

$$E_C = n(E - TS + pV - \Omega J). \quad (5.12)$$

Using like above tracelessness of the stress tensor, $pV = E/n$, we get

$$E_C = \frac{nV_n}{8\pi G} r_+^{n-3} (r_+^2 + a^2) = 2E. \quad (5.13)$$

This, together with the thermodynamical quantities (5.10), satisfies again the Cardy-Verlinde formula (5.6). If we write (5.6) in the form (3.6) with $L_0 = Er_+$ and $c/6 = 2E_C r_+$, we see that the central charge c is equal to $6nS/\pi$. Monotonicity of this generalized c-function with respect to temperature changes is thus equivalent to positive specific heat, i. e. , to thermodynamic stability. Of course for the Schwarzschild black hole (5.1) the specific heat is always negative. For the Kerr solution (5.7) it is straightforward to show that the specific heat $C_J = (\partial S/\partial T)_J$ is positive for $n = 2$, $1 < r_+^2/a^2 < 3 + 2\sqrt{3}$, and for $n = 3$, $0 < r_+^2/a^2 < 3$. For $n \geq 4$ there are no regions of positive C_J .

As a final remark we would like to emphasize that in the above considerations of asymptotically flat Schwarzschild and Kerr black holes in $n + 2$ dimensions, we explicitly assumed tracelessness of the stress tensor of a hypothetical underlying field theory in $n + 1$ dimensions. We then found that a generalized Cardy-Verlinde formula holds. This suggests that, like their cousins in AdS space, also the black holes (5.1) and (5.7) may admit a dual description in terms of a conformal field theory that lives in one dimension lower.

In this context it is interesting to note that the central charge $c = 6nS/\pi$ found above is proportional to $(r_+/l_P)^n$, where l_P denotes the Planck length in $n + 2$ dimensions. Due to the holographic principle [25, 26] there should be one degree of freedom per Planck volume, so c represents the total number of degrees of freedom on the event horizon, and thus makes indeed sense as a central charge. Note that for the Schwarzschild black hole, one has

$$E = \frac{c}{6} \pi^2 r_+ \left(\frac{2T}{n-1} \right)^2, \quad (5.14)$$

which is exactly the energy-temperature relation of a two-dimensional CFT with characteristic length r_+ and effective temperature $\tilde{T} = 2T/(n-1)$. Alternatively, one can write (5.14) in the form

$$Er_+ = \frac{c}{24}, \quad (5.15)$$

which is the ground state energy of a CFT in two dimensions.

6 Conclusions

In the present paper we analyzed the thermodynamics of conformal field theories with AdS duals, and showed that they share a completely universal behavior, in that a generalized Cardy formula is valid even when R-charges or bulk supergravity scalars are turned on, or when the CFT resides on a rotating spacetime. We argued that the validity of the Cardy-Verlinde formula can be traced back to the fact that these CFTs share many features in common with conformal field theories in two dimensions. For instance, they satisfy the general relation

$$2ER = \frac{n}{b\pi} S_C [1 + \Delta^{-2}], \quad (6.1)$$

where the function Δ encodes the detailed properties of the conformal field theory, e. g. value of the coupling constant, presence of R-charges, VEVs of certain operators or rotation.

Much of our intuition has been gained by the fact that we were able to write the Casimir entropy S_C like a two-dimensional central charge. Then it was natural to think of S_C as a generalized C -function. In this spirit we have studied the behavior of S_C both under temperature changes and when non-trivial bulk fields are turned on. The idea was to see if the presence of non-trivial fields in the bulk theory were to induce a holographic RG-flow in the dual field theory. The examples studied here support such an interpretation. Indeed the Casimir entropy S_C , which in general is a function of the temperature and of the renormalization scale of the field theory, is a good candidate for a generalized C -function for field theories that have a dual supergravity description, both at zero and at finite temperature. We have found that S_C is a monotonic function in each variable, separately. Keeping the energy scale fixed we move in the space of conformally invariant theories at various temperatures; if we fix the temperature and vary the energy scale we move along the renormalization group trajectories. It becomes extremely interesting to study the behavior of S_C in complete generality.

For $\mathcal{N} = 4$, $D = 4$ SYM theory, one finds $b = 2$ at strong coupling and $b = 4$ at weak coupling, suggesting that a generalized Cardy formula may hold at all couplings, with b a function of $g_{YM}^2 N$, interpolating smoothly (for temperatures above the Hawking-Page phase transition) between the two regimes $g_{YM}^2 N \rightarrow 0$ and $g_{YM}^2 N \rightarrow \infty$. It would also be of interest to test our idea by explicitly calculating corrections to the Cardy-Verlinde entropy formula both at weak and at strong coupling.

We did not succeed to show the validity of a generalized Cardy formula for the $(0, 2)$ free CFT in six dimensions or for the $\mathcal{N} = 8$ supersingleton theory in three dimensions. The fact that such a formula holds for the weakly coupled $D = 4$, $\mathcal{N} = 4$ SYM model, but not for the other CFTs with AdS duals (at weak coupling), may be connected to the non-renormalization properties of the former theory and deserves further study.

Acknowledgments

D. K., A. C. P. and D. Z. are partially supported by INFN, MURST and by the European Commission RTN program HPRN-CT-2000-00131, in which they are associated to the University of Torino. G. S. is supported by the US Department of Energy under grant DE-FG05-91ER40627.

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